set

A set is an unordered collection of distinct objects called elements or members of the set. The notation $a \notin A$ denotes that a is not an element of the set A. The notation $a \notin A$ denotes that a is not an element of the set A.

eg: set of all vowels in the English alphabet

* Two sets are equal iff they have the same elements.

The size of a set

Let S be a set, if there are exactly n distinct elements in S where n is a nonnegative integer. we say that S is a finite set and that n is the cardinality of S.

power set

Given a set S, the power set of S is the set of all subsets of the set S. The power set of S is denoted by P(S).

eg: Let $A = \{1, 2\}$

$P(A) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$

Cartesian products

Let A and B be sets. The cartesian product of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence

$$A \times B = \{ (a,b) \mid a \in A \wedge b \in B \}$$

Eg: $A = \{1, 2, 3\}$, $B = \{4, 5\}$

$$A \times B = \{ (1,4), (1,5), (2,4), (2,5), (3,4), (3,5) \}$$

Set operation

$$A \cup B = \{ x \mid x \in A \vee x \in B \}$$

$$A \cap B = \{ x \mid x \in A \wedge x \in B \}$$

* Two sets are called disjoint if their intersection is the empty set.

$$A - B = \{ x \mid x \in A \wedge x \notin B \}$$

$$\bar{A} = \{ x \in U \mid x \notin A \}$$

Relations

* Let A and B be sets. A binary relation from A to B is a subset of $A \times B$.

* A relation on a set A is a relation from A to A

eg: $A = \{1, 2, 3, 4\}$

$$R = \{ (1,1), (1,2), (1,3), (1,4), (2,2) \}$$

? How many relations are there on a set with n elements?.

A). A relation on a set A is a subset of AXA. Because AXA has n^2 elements when A has n elements, and a set with m elements has 2^m subsets, there are 2^{n^2} subsets of AXA. Thus, there are 2^{n^2} relations on a set with n elements.

properties of Relations

* A relation R on a set A is called reflexive if $(a,a) \in R$ for every element $a \in A$.

Eg: consider the following relations on $\{1,2,3,4\}$

$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$

$R_2 = \{(1,1), (1,2), (2,1)\}$

$R_3 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)\}$

$R_4 = \{(2,1)(3,1), (3,2), (4,1), (4,2), (4,3)\}$

$R_5 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$

$R_6 = \{(3,4)\}$

Reflexive relations, R_3, R_5

* A relation R on a set A is called symmetric

if $(b,a) \in R$ whenever $(a,b) \in R$ $\forall a,b \in R$.

A relation R on a set A such that $\forall a,b \in A$ if $(b,a) \in R$ ~~then and~~ / if $(a,b) \in R$ then $a=b$ is called antisymmetric.

Eg: The equality relation is symmetric bcoz, $a=b$ iff $b=a$.

Eg2: The less than or equal to relation is antisymmetric

? Is the "divides" relation on the set of +ve integers symmetric? Is it antisymmetric?

A). This relation is not symmetric because $1|2$ but $2 \nmid 1$. However, it is antisymmetric ~~To see th~~

* A relation R on a set A is called transitive if whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$, $\forall a,b \in A$

$(\forall a \forall b \forall c ((a,b) \in R \wedge (b,c) \in R) \to (a,c) \in R$

? Is this "divides" relation on the set of positive integers transitive?

A). Suppose that a divides b and b divides c. Then there are positive integers k and l such that ~~a=bk~~, $b = ak$ and $c = bl$ Hence, $c = (ak)l$
$= a(kl)$,
so a divides c. It follows that this relation is

transitive.

? How many reflexive relations are there on a set with n elements?

A A relation R on a set A is a subset of $A \times A$. consequently, a relation is determined by specifying whether each other of the n^2 ordered pairs in $A \times A$ is in R. However, if R is reflexive, each of the n ordered pairs (a,a) for $a \in A$ must be in R. Each of the other $n(n-1)$ ordered pairs of the form (a,b), wher $a \neq b$, may ~~or~~ may not be in R, hence by the product rule for counting, there are $2^{n(n-1)}$ reflexive relations [this is the no. of ways to choose whether each element (a,b), with $a \neq b$, belongs to R].

Equivalence relations

* A relation on a set A is called an equivalance relation if it is reflexive, symmetric and transitive
* Two elements a and b that are related by an equivalance relation are called equivalent. The notation $a \sim b$ is often used to denote that a & b are equivalent elements with respect to a particular equivalence relation.

eg. Let R be the relation on the set of integers such that aRb iff $a = b$ or $a = -b$

? Let R be the relation on the set of real numbers such that aRb iff a-b is an integer is R an equivalence relation?

A). Because $a-a=0$ is an integer for all real numbers a, aRa $\forall$ real numbers a. Hence, R is reflexive. Now suppose that aRb. Then a-b is an integer, so b-a is also an integer. Hence, bRa. It follows that R is symmetric. If aRb and bRc, then a-b & b-c are integers. Therefore, $a-c=(a-b)+(b-c)$ is also an integer. Hence aRc. Thus R is transitive. consequently, R is an equivalence relation

imp

? congruence Modulo m. Let m be an integer with $m>1$. S.T the relation $R=\{(a,b) \mid a \equiv b \pmod m\}$ is an equivalence relation on the set of integers.

A). $a \equiv b \pmod m$ iff m divides a-b. $a-a=0$ is divisible by m. bcoz $0=0 \cdot m$. Hence $a \equiv a \pmod m$, so congruence modulo m is reflexive.

Now suppose that $a \equiv b \pmod m$. Then a-b divisible by m. so $a-b=km$, where k is an integer. It follows that $b-a=(-k)m$, so $b \equiv a \pmod m$. Hence, congruence modulo m is symmetric.

suppose that $a \equiv b \pmod m$ & $b \equiv c \pmod m$. Then m divides bot $(a-b)$ & $(b-c)$. $\therefore$ there are integers k & l such that $a-b = km$ & $b-c = lm$

$\therefore$ $a-b+b-c = (k+l)m$

$a-c = (k+l)m$

$\Rightarrow$ $a \equiv c \pmod{m}$

$\therefore$ congruence modulo m is transitive.

It follows that congruence modulo m is equivalance relation.

? Consider the set Z of integers. Define aRb if there is a positive integer r such that $b = a^r$. For instance, $2R8$ since $8 = 2^3$, Then R is a partial ordering of Z

A) Partial ordering relation.

A relation on a set A is called partial ordering if it is reflexive, antisymmetric & transitive.

A) Consider the relation $b = a^r$, r is a +ve integer.

Reflexive

$aRa \Rightarrow$

$a = a$

$a = a^1 \Rightarrow aRa \Rightarrow a = a^1$

$\therefore$ R is reflexive

Transitive

Let $a, b, c \in Z$

aRb & $bRc \Rightarrow b = a^r$ —— (1)

$c = b^s$ —— (2)

(1) in (2) $\Rightarrow$

$c = a^{rs}$

$\Rightarrow$ aRc

$\therefore$ R is transitive.

antisymmetric

Let $a, b \in Z$

$aRb \Rightarrow b = a^r$ ——— (3)

$bRa \Rightarrow a = b^s$ ——— (4).

(3) in (4) $\Rightarrow$

$a = (a^r)^s$

$a = a^{rs} \Rightarrow rs = 1$

$\Rightarrow r = 1$ & $s = 1$

$r = 1$ in (3)

$b = a$

$\therefore$ R is antisymmetric.

2 Let A be a set of nonzero integers and let $\approx$ be the relation on $A \times A$ defined by $(a,b) \approx (c,d)$ whenever $ad = bc$. prove that $\approx$ is an equivalence relation.

A Give that A be a set of nonzero integers. Let $\approx$ be the relation on $A \times A$ defined by, $(a,b) \approx (c,d)$ whenever $ad = bc$

Let $(a,b) \in A \times A$ $\Rightarrow$

$(a,b) \in A \times A$

$ab = aba$ ($\because$ $\cancel{ad = bc}$)

$\therefore (a,b) \approx (a,b)$.

$\therefore \approx$ is reflexive relation

<u>Symmetric</u>

$(a,b), (c,d) \in A \times A$ & $(a,b) \approx (c,d)$

$\Rightarrow ad = bc$

Let $(c,d), (a,b) \in A \times A$ and we've to prove that

~~$(b,a) \approx (d,c)$~~. $(c,d) \approx (a,b)$

We know $ad = bc \Rightarrow da = cb \Rightarrow (c,d) \approx (a,b)$

$\Rightarrow$ [illegible]

$\Rightarrow$ [illegible]

$\therefore \approx$ is symmetric.

Transitive

$ad = bc$

$cf = de$

Let $(a,b), (c,d), \in A \times A$ & $(a,b) \approx (c,d)$ & $(c,d) \approx (e,f)$.

Let (e,f)

$(a,b) \approx (c,d) \Rightarrow ad = bc$ ——— (1)

$(c,d) \approx (e,f) \Rightarrow cf = de$ ——— (2).

we've to show that, $(a,b) \approx (e,f)$, ie, $af = be$

from (1), we get

$d = \frac{bc}{a}$, substitut thes value in (2).

$\Rightarrow cf = \frac{bc}{a} e$

$\Rightarrow af = be$

$\Rightarrow (a,b) \approx (e,f)$

$\therefore \approx$ is transitive.

Q. Give an example of a relation R on $A = \{1, 2, 3\}$ such that:

(a) R is both symmetric & antisymmetric

(b) R is neither symmetric nor antisymmetric

A). $A = \{1, 2, 3\}$

a) $R_1 = \{(1,1), (2,2)\}$

b) $R_2 = \{(1,2), (2,3), (2,1)\}$

Combining Relations

Because relations from A to B are subsets of $A \times B$, two relations from A to B can be combined in any way two sets can be combined.

Q. Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. The relation $R_1 = \{(1,1), (2,2), (3,3)\}$ and $R_2 = \{(1,1), (1,2), (1,3), (1,4)\}$ can be combined to obtain

$R_1 \cup R_2 = \{(1,1), (2,2), (1,2), (1,3), (1,4), (3,3)\}$

$R_1 \cap R_2 = \{(1,1)\}$

$R_1 - R_2 = \{(2,2), (3,3)\}$

$R_2 - R_1 = \{(1,2), (1,3), (1,4)\}$

$A \oplus B = A \cup B - A \cap B$

2 Let R_1 & R_2 be the set less than relation on the set of real numbers and R_2 be the greater than relation on the set of all real numbers. ie, $R_1 = \{(x,y) : x<y\}$ and $R_2 = \{(x,y) : x>y\}$. what are $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, $R_2 - R_1$ & $R_1 \oplus R_2$.

A). we note that $(x,y) \in R_1 \cup R_2$ iff $(x,y) \in R_1$ or $(x,y) \in R_2$. Hence, $(x,y) \in R_1 \cup R_2$ iff $x<y$ or $x>y$. Because the condition $x<y$ or $x>y$ is the same as the condition $x \neq y$, it follows the that $R_1 \cup R_2 = \{(x,y) : x \neq y\}$. in other words, the union of the less than relation and the greater than relation is the not equals relation.

Next, note that it is impossible for a pair (x,y) to belong to both R_1 & R_2 bcoz it is impossible that $x<y$ and $x>y$. It follows that $R_1 \cap R_2 = \phi$. we also see that $R_1 - R_2 = R_1$. $R_2 - R_1 = R_2$, and $R_1 \oplus R_2 = R_1 \cup R_2 - R_1 \cap R_2 = \{(x,y) : x \neq y\}$

2. If R_1 and R_2 are equivalence relations on a set A, show that $R_1 \cap R_2$ is also an equivalence relation.

A). Suppose that R_1 and R_2 are two equivalence relations on a non empty set A.
First we prove that $R_1 \cap R_2$ is reflexive.

(i) $R_2 \cap R_2$ is reflexive

Let $a \in A$ arbitrarily

Then $(a,a) \in R_1$ and $(a,a) \in R_2$ ($\because R_1, R_2$ both are reflexive)

So $(a,a) \in R_1 \cap R_2$

$\Rightarrow R_1 \cap R_2$ is reflexive.

(ii) $R_1 \cap R_2$ is symmetric

Let $a,b \in A$ such that $(a,b) \in R_1 \cap R_2$

$\therefore (a,b) \in R_1$ and $(a,b) \in R_2$

$\Rightarrow (b,a) \in R_1$ and $(b,a) \in R_2$.

($\because R_1, R_2$ both are symmetric)

$\therefore R_1 \cap R_2$ is a symmetric relation.

(iii) $R_1 \cap R_2$ is transitive

Let $a,b,c \in A$ such that $(a,b) \in R_1 \cap R_2$ and $(b,c) \in R_1 \cap R_2$.

$\Rightarrow (a,b) \in R_1 \cap R_2 \Rightarrow (a,b) \in R_1$ and $(a,b) \in R_2$ —(1)

$(b,c) \in R_1 \cap R_2) \Rightarrow (b,c) \in R_1$ and $(b,c) \in R_2$ —(2)

(1) & (2) $\Rightarrow (a,b)$ and $(b,c) \in R_1$

$\Rightarrow (a,c) \in R_1$, since R_1 being an equivalence relation is also transitive.

similarly, we can prove that $(a,c) \in R_2$

$\therefore (a,c) \in R_1 \cap R_2$, so, $R_1 \cap R_2$ is transitive.

Thus $R_1 \cap R_2$ is reflexive, symmetric and also transitive. Thus $R_1 \cap R_2$ is an equivalence relation.

? Let R_1 and R_2 be equivalence relations on a set A, then $R_1 \cup R_2$ may or may not bge transitive.

A). If R_1 and R_2 are transitive on a set A then $R_1 \cup R_2$ may or maynot be transitive

consider the set $A = \{1,2,3\}$

Let $R_1 = (1,1), (2,2), (3,3), (1,2), (2,1)$

and $R_2 = (1,1), (2,2), (3,3), (1,3)\ (3,1)$ then the R_1 and R_2 are equivalence relations on A. However,

$R_1 \cup R_2 = (1,1), (2,2), (3,3), (1,2), (2,1), (1,3), (3,1)$ is not transitive as $(2,1)$ and $(1,3) \in R_1 \cup R_2$ but $(2,3) \notin R_1 \cup R_2$.

Representing Relations using Matrices

A relation b/w finite sets can be represented using a zero-one matrix.

suppose that R is a relation from $A = \{a_1, a_2 \ldots a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$. The relation R can be represented by the matrix $M_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

2 suppose that $A = \{1,2,3\}$ and $B = \{1,2\}$. Let R be the relation from A to B containing (a,b) if $a \in A$, $b \in B$ and $a > b$. What is the matrix

representing R if $a_1 = 1$, $a_2 = 2$, & $a_3 = 3$ and $b_1 = 1$ and $b_2 = 2$?

A). $R = \{(2,1), (3,1), (3,2)\}$ the matrix for R is,

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \end{array}$$

? Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$ which ordered pairs are in the relation R represented by the matrix.

$$M_R = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array}\begin{array}{c} \begin{array}{ccccc} b_1 & b_2 & b_3 & b_4 & b_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array} ?$$

A. $R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), (a_3, b_3), (a_3, b_5)\}$.

* The matrix of a relation on a set, which is a square matrix can be used to determine whether the relation has certain properties.

a) R is a relation on a set A, R is reflexive if all the elements iff $m_{ii} = 1$, for $i = 1, 2, \ldots n$.

In other words, R is reflexive if all the elements on the diagonal ~~can be~~ of M_R are equal to 1.

b) The relation R is symmetric iff $m_{ji} = 1$ whenever $m_{ij} = 1$. This is also means $m_{ji} = 0$ whenever $m_{ij} = 0$. consequently, R is symmetric iff $m_{ji} = m_{ij} \ \forall (i,j)$, $i = 1,2, \dots n$

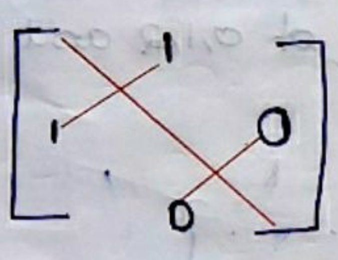
$, j = 1,2, \dots n$

* R is symmetric iff M_R is a symmetric matrix.

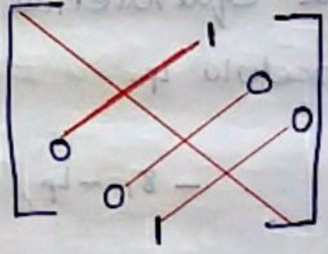
ie, $M_R = (M_R)^t$

c) The ~~rel~~ relation R is antisymmetric, it has the property that if $m_{ij} = 1$ with $i \neq j$ then $m_{ji} = 0$ or, in other words, either $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$

Eg:



a) symmetric



b) antisymmetric

2 suppose that the relation R on a set is represented by the matrix.

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Is R is reflexive, symmetric and/or antisymmetric?

A). Because all the digonal elements of the matrix are equal to 1, R is reflexive, moreover Because M_R is symmetric, it follows that R is symmetric. it is also easy to see that R is not antisymmetric.

Equivalence classes

Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the equivalence class of a. The equivalence class of a with respect to R is denoted by $[a]_R$. ie, $[a] = \{x \in A \mid aRx\}$

* If $b \in [a]_R$, then b is called a representative of this equivalence class.

? What are the equivalence classes of 0,1,2 and 3 for congruence modulo 4.

A).

$$[0] = \{ \ldots, -8, -4, 0, 4, 8, \ldots \}$$

$$[1] = \{ \ldots, -7, -3, 1, 5, 9, \ldots \}$$

$$[2] = \{ \ldots, -6, -2, 2, 6, 10, \ldots \}$$

$$[3] = \{ \ldots, -5, -1, 3, 7, 11, \ldots \}$$

* Note that every integer is in exactly one of the four equivalence class and that the integer n is in the class containing n mod 4.

* Let R be an equivalence relation on a set S. Then the collection of all such equivalence classes in S under R is a partition of S.

specifically,

i) for each a in S, we have $a \in [a]$

ii) $[a] = [b]$, iff $(a,b) \in R$ (ie, aRb)

iii If $[a] \neq [b]$, then $[a]$ and $[b]$ are disjoint

proof,

since R is an equivalence relation on S, it is reflexive symmetric and transitive.

i) since R is reflexive, for each $a \in S$, $(a,a) \in R$ ie, aRa. Hence, from the definition of equivalence class of a, it follows that $a \in [a]$, $\forall a \in S$.

ii) for any $a,b \in S$, first let $(a,b) \in R$. Then we've to prove that $[a] = [b]$

choose any $x \in [b]$, Then, by the definition of equivalence class of, $(b,x) \in R$ (bRx). By hypothesis, $(a,b) \in R$. Since R is transitive

$(a,b), (b,x) \in R \Rightarrow (a,x) \in R$

$\Rightarrow aRx$

Hence $x \in [a]$. since $x \in [b]$ implies $x \in [a]$,

$[b] \subseteq [a]$.

choose any $x \in [a]$. Then, by the definition of equivalence class of a, $(a,x) \in R$ (ie, aRx)

~~since R~~ By hypothesis, $(a,b) \in R$ and hence by symmetry of R, $(b,a) \in R$. since R is transitive $(b,a), (a,x) \in R \Rightarrow (b,x) \in R$. Hence $x \in [b]$. since $x \in [a] \Rightarrow x \in [b]$, $[a] \subseteq [b]$

$[a] \subseteq [b]$ and $[b] \subseteq [a] \Rightarrow [a] = [b]$

conversly,

Let $[a] = [b]$, $[a] = [b] \Rightarrow b \in [b] = [a]$

$\Rightarrow b \in [a]$

$\Rightarrow aRb$

$\Rightarrow (a,b) \in R$

iii) Let $a, b \in S$ and $[a] \neq [b]$. Then we have to prove that $[a]$ and $[b]$ are disjoint.

If possible, let $[a]$ and $[b]$ are not disjoint

ie, $[a] \cap [b] \neq \phi$

choose any $x \in [a] \cap [b]$

$x \in [a] \cap [b] \Rightarrow x \in [a]$ and $x \in [b]$

$\Rightarrow (a,x) \in R$ and $(b,x) \in R$

$\Rightarrow (a,x) \in R$ and $(x,b) \in R$

(By symmetric property of R)

$\Rightarrow (a,b) \in R$

(By transitivity of R)

$\Rightarrow [a] = [b]$ (by (ii))

This is a contradiction. Hence our assumption that [a] and [b] are not disjoint, is wrong. $\therefore$ [a] and [b] are disjoint.

Note:

* If $A = \{1,2,3,4,5\}$ and $R = \{(1,1),(2,2),(2,3),(3,2),(3,3),(4,4),(4,5),(5,4),(5,5)\}$, then R is an equivalence relation on A. Here $[1] = \{1\}$, $[2] = \{2,3\} = [3]$, $[4] = \{4,5\}$ and $[5] = \{4,5\}$ and,

$A = [1] \cup [2] \cup [4]$ with $[1] \cap [2] = \phi$, $[1] \cap [4] = \phi$ and $[2] \cap [4] = \phi$. So $\{[1],[2],[4]\}$ determines a partition of A.

* If an equivalence relation R on $A = \{1,2,3,4,5,6,7\}$ induces the partition

$A = \{1,2\} \cup \{3\} \cup \{4,5,7\} \cup \{6\}$, what is R?

consider the cell $\{1,2\}$ of the partition. This subset implies that $[1] = \{1,2\} = [2]$ and so $(1,1),(2,2),(1,2),(2,1)$ ~~of H~~ $\in R$ (The first two ordered pairs are necessary for the reflexive property of R. the others preserve symmetry)

In like manner, the cell $\{4,5,7\}$ implies that under R, $[4] = [5] = [7] = \{4,5,7\}$ and that, as an equivalence relation, R must contain $\{4,5,7\} \times \{4,5,7\}$. In fact,

$R = (\{1,2\} \times \{1,2\} \cup \{3\} \times \{3\} \cup \{4,5,7\} \times \{4,5,7\} \cup \{6\} \times \{6\})$

Composition of Relations

Let A, B, C be sets and R be a relation from A to B and let S be a relation from B to C. Then R and S give rise to a relation from A to C denoted by RoS and defined as follows:

$$RoS = \{ (a,c) : \exists\, b \in B \text{ for which } (a,b) \in R \text{ and } (b,c) \in S \}$$

The relation RoS is called the composition of R and S.

Eg: Let $A = \{a, b, c, d, e\}$, $B = \{1, 2, 3, 5\}$,
$C = \{0, 4, 7, 8\}$ and $R=$

$R = \{ (a,2), (a,5), (b,3), (c,1), (c,3), (d,5) \}$

$S = \{ (1,7), (2,0), (2,7), (5,0), (5,4) \}$

then

$RoS = \{ (a,0), (a,7), (a,4), (c,7), (d,0), (d,4)$

---

* Composition Relation is satisfy the associative law.

? Show that a relation R on a set A is transitive iff $R^2 \subseteq R$

A). Let R be a relation on a set A, first let R is transitive and we've to show that R^2 (RoR) $\subseteq R$.

Let $(x,y) \in R^2$

from the definition of composition relation, $R \circ R$, there exists an element $z \in A$ such that (x,z) & $(z,y) \in R$

(x,z) & $(z,y) \in R \Rightarrow (x,y) \in R$

(since R is transitive).

$\therefore (x,y) \in R^2 \Rightarrow (x,y) \in R$

$\Rightarrow R^2 \subseteq R$

conversely, let us assume that $R^2 \subseteq R$ and we have to show that R is transitive. that is, for any, $x,y,z \in A$ if $(x,y), (y,z) \in R$ then ~~case~~ $(x,z) \in R^2$. since $R^2 \subseteq R$, we get $(x,z) \in R$

Thus for any $x,y,z \in A$, $(x,y), (y,z) \in R$ implies $(x,z) \in R$. Hence R is transitive.

2. Let f be a relation from R^2 to R^2 defined as $f = \{((x,y),(a,b)) : x^2+y^2 = a^2+b^2\}$ show that f is an equivalence relation?

A). suppose that $f = \{(x,y),(a,b) : x^2+y^2 = a^2+b^2\}$.

first, we have to show that f is reflexive.

Reflexive

Let $(x,y) \in R^2$ and $x^2+y^2 = x^2+y^2$, clearly

$((x,y),(x,y)) \in f$

$\therefore f$ is Reflexive.

Symmetry

Let $(x,y), (a,b) \in R^2$ and we have to show that, $((a,b),(x,y)) \in f$ if $((x,y),(a,b)) \in f$

Let $((x,y),(a,b)) \in f$, then from the definition of f, we have,

$$x^2+y^2 = a^2+b^2$$

$$\Rightarrow \cancel{y^2+x^2}\ a^2+b^2 = x^2+y^2$$

$$\Rightarrow ((a,b),(x,y)) \in f$$

$\therefore$ f is symmetric.

Transitive

Let $((x,y),(a,b)) \in f$ and $((a,b),(d,f)) \in f$ then we've to show that $((x,y),(d,f)) \in f$

$((x,y),(a,b)) \in f \Rightarrow x^2+y^2 = a^2+b^2$ ——— (1)

$((a,b),(d,f)) \in f \Rightarrow a^2+b^2 = d^2+f^2$ ——— (2)

from (1) and (2) we get

$x^2+y^2 = d^2+f^2$

$\Rightarrow ((x,y),(d,f)) \in f$

$\therefore$ f is transitive, and hence f is an equivalence relation

Theorem

Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S. conversely, given a partition $\{A_i \mid i \in I\}$ of the set S, there is an equivalence relation R that has the set $A_i, i \in I$, as its equivalence classes.

* There is another way of finding the composition $R \circ S$ of relations using matrices. Let M_R, M_S and $M_{R \circ S}$ denote respectively the matrices of the relations R, S and $R \circ S$ given above, Then

$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \qquad M_S = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 4 & 7 & 8 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

and

$$M_{R \circ S} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 4 & 7 & 8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Multiplying M_R and M_S we obtain

$$M_R M_S = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

* Transitivity condition in matrix representation.

$$M_R^2 + M_R = M_R$$

2 Let R be a relation on a set $A = \{1,2,3,4\}$
The matrix representation of R is given by,

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

check whether R is transitive or not?

A). $M_R^2 = M_R \cdot M_R$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_R^2 + M_R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} = M_R$$

$\therefore$ R is transitive.

* $R^{-1} = M_R^T$

* $R^c = M_R^c$

Closures of Relations

Reflexive closure

The reflexive closure of a binary relation R on a set A is the smallest reflexive relation on A that contains R.

Eg: suppose that R is the Relation,

$R = \{(1,1), (1,2), (2,1), (3,2)\}$ on ~~the~~ the set A

clearly R is not reflexive.

$R^{(r)} = \{(1,1), (1,2), (2,2), (2,1), (3,3), (3,2)\}$

Note:

Let $R = \{(1,1), (1,4), (2,2)\}$

and $R^{(r)} = \{(1,1), (2,2), (3,3), (4,4), (1,4), (2,4)\}$

then $R^{(r)}$ is not a reflexive clousure of R.

Symmetric closure.

The symmetric closure of a binary relation R on a set A is the smallest symmetric relation on A that contains R.

Eg: let $R = \{(1,1), (1,2), (2,2), (2,3), (3,1), (3,2)\}$ on $\{1,2,3\}$ is not symmetric.

$R^{(s)} = \{(1,1), (1,2), (2,1), (2,2), (2,3), (3,2), (3,1), (1,3)\}$

Transitive closure

The ~~relat~~ transitive closure of a binary relation R on a set A is the smallest transitive relation on A that contains R.

* Let R be a relation on a set A with n elements then transitive closure of R is given by,

$$R^{(t)} = R \cup R^2 \cup \cdots \cup R^n$$

Eg: Let $A = \{1, 2, 3\}$ and R be a relation on A,

$$R = \{(1,2), (2,3), (3,1)\}$$

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$M_{R^2} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$M_{R^3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\cancel{M_R + M_{R^2} +}$ $M_R + M_{R^2} + M_{R^3} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \underline{\underline{R^{(t)}}}$

Warshall's Algorithm

* Let R be a relation on a set A with n elements then transitive closure of R is given by,

$R^{(t)} = R \cup R^2 \cup \cdots \cup R^n$

Eg: Let $A = \{1,2,3\}$ and R be a relation on A,

$R = \{(1,2), (2,3), (3,1)\}$

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$M_R^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$M_R^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\cancel{M_R + M_R^2 +}$ $M_R + M_R^2 + M_R^3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = R^{(t)}$

Warshall's Algorithm

```
procedure Warshall ( MR : n x n zero-one matrix)
W := MR
for k := 1 to n
    for i := 1 to n
        for j := 1 to n
        Wij := Wij ∨ (Wik ∧ Wkj)
```

return W { $W = [W_{ij}]$ is $M_{R^{(t)}}$ }

2. Let R be the relation ~~with~~ on the set $A = \{1, 2, 3, 4\}$, R is given below,

$$R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

N. W_0 is the matrix of the relation, hence,

$$W_0 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$W_1 = C_1 R_1$

$$\omega_1 = \begin{bmatrix} 1 & - & - & 1 \\ 1 & 1 & - & - \\ - & - & - & 1 \\ - & - & - & 1 \end{bmatrix}$$

$C_1 \to 1, 2$

$R_1 \to 1$

ordered pairs,

$(1,1)$ $(2,1)$

~~er~~

$$W_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

W_2 $(C_2 R_2)$

$$W_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to 2$

$R_2 \to 1, 2$

$(2,1), (2,2)$.

$W_3 (C_3 R_3)$

$$W_3 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 \rightarrow X$

$$W_4 = R^{(t)} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_4 R_4$

$C_4 \rightarrow 3, 4$

$R_4 \rightarrow 4$

$(3,4), (4,4)$

2. Let $A = \{1, 2, 3\}$ and $R = \{(1,2), (2,3), (3,3)\}$

A).
$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = W_0$$

$W_1 \rightarrow C_1 \rightarrow X$

R_1

$$W_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$W_2 (C_2, R_2) \rightarrow C_2 \rightarrow 1, R_2 \rightarrow 3$

$\Rightarrow (1, 3)$

$$W_2 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$W_3 (C_3, R_3)$

$C_3 \rightarrow 1, 2, 3$

$R_3 \rightarrow 3 \Rightarrow (1,3), (2,3), (3,3)$

$$W_3 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = R^{(t)}$$

Note:

If R is ~~the number~~ an $n \times n$ matrix then W_n is the transitive closure of R.

n-ary Relation.

Let $A_1, A_2, \dots A_n$ be sets. An n-ary relation on these sets is a subset of $A_1 \times A_2 \times \dots \times A_n$. The sets $A_1, A_2 \dots A_n$ are called the domains of the relation, and n is called the degree.

Eg: Let R be the relation on $N \times N \times N$ consisting of triples (a, b, c), where a, b and c are integers with $a < b < c$. Then $(1, 2, 3) \in R$, but $(2, 4, 3) \notin R$. The degree of this relation is 3. Its domains are all equal to the set of natural numbers.

Eg2: Let R be the relation on $Z \times Z \times Z^+$ consisting of triples (a, b, m), where a, b and m are integers with $m \geq 1$ and $a \equiv b \pmod m$. Then $(8, 2, 3)$, $(-1, 9, 5)$ and $(14, 0, 7)$ all belong to R, but $(7, 2, 3)$ $(-2, -8, 5)$ and $(11, 0, 6)$ do not belong to R bcoz $8 \equiv 2 \pmod 3$, $-1 \equiv 9 \pmod 5$ and $14 \equiv 0 \pmod 7$, but $7 \not\equiv 2 \pmod m$, $-2 \not\equiv -8 \pmod 5$, and $11 \not\equiv 0 \pmod 6$. This relation has degree 3 and its first two domains are the set of all integers and its third domain is the set of positive integers.

function

Let X and Y ~~are~~ be any two non empty sets. A function or mapping from X to Y is a rule that assigns to each element in X a unique element in Y. If f denotes these assignments we write

$f : X \to Y$

which reads "f is a function from X to Y".

* The set X is called the domain of the function f and Y is called the co-domain of f.

* set consisting ~~of~~ precisely of those elements in Y which appear as the image of atleast one element in X is called the range or image of f.

* we usually denote the domain of f by D(f) and the range of f by Ran(f) or Im(f) or f(x)

ie,

$Im(f) = \{ y \in Y : y = f(x) \text{ for some } x \in X \}$.

$Im(f) \subseteq Y$

- Eg: consider the function $f(x) = x^3$. ie, f. assigns to each real number its cube. Then the image of 2 is 8 and so $f(2) = 8$ similarly $f(3) = 27$, and so on.

* Let f_1 and f_2 be functions from A to R. Then f_1+f_2 and f_1f_2 are also functions from A to R. defined $\forall x \in A$ by,

$$(f_1+f_2)(x) = f_1(x) + f_2(x)$$
$$(f_1f_2)(x) = f_1(x) \cdot f_2(x).$$

one-to-one function.

A function f is said to be one-to-one or an injection, iff $f(a)= f(b)$ implies that $a=b$ for all a and b in the domain of f.

Q. Determine whether the function $f(x) = x+1$ from the set of real numbers to itself is one-to-one.

A) suppose that x and y are real numbers with

$f(x)= f(y)$

$\Rightarrow x+1 = y+1$

$\Rightarrow x = y$

hence $f(x) = x+1$ is a 1-1 function from $R \to R$

Q. Determine whether the function $f(x)=x^2$ from the set of integers to the set of integers is one-one.

A). Here. $f: Z \to Z$ and $f(x)=x^2$

suppose that x and y are two integers such that

$f(x)= f(y)$

$\Rightarrow x^2 = y^2$.

This is not true in all case. for example $f(1)=f(-1)$

but $1 \neq -1$,

$\therefore$ f is not 1-1

Onto function.

A function from A to B is called onto, or a surjection, iff for every element $y \in B$ there is a element $x \in A$ with. $f(x) = y$.

Eg: $f(x) = x+1$ $(f: R \to R)$

Let y is a real number

$y = x+1$ ——(i)

$x = y-1$

put $x = y-1$ in (i).

$y = y-1+1$ ~~y=1~~

$= y$ //

$\therefore$ ~~f is onto.~~

$y-1$ is also a real number and so, is in the domain of f. and $f(y-1) = y$, where $y \in R$.

$\therefore$ f is onto.

Bijective function.

The function f is a bijection, if it is both one-on & onto.

eg.. consider the function $f: R \to R$: $f(x) = 2x+3$

for any $x, y \in R$

$f(x) = f(y) \Rightarrow 2x+3 = 2y+3$

$\Rightarrow 2x = 2y$

$\Rightarrow x = y$

hence f is one - one.

For any $y \in R$, $\frac{y-3}{2} \in R$. $y = 2x+3$, $\frac{y-3}{2} = x$

and $f\left(\frac{y-3}{2}\right) = 2\left(\frac{y-3}{2}\right) + 3$

$$= y - 3 + 3$$
$$= y \,//$$

Thus for all $y \in R$ there exist $x = \frac{y-3}{2} \in R$ such that

$$f(x) = f\left(\frac{y-3}{2}\right) = y \,//$$

hence f is onto.

$\therefore$ It is a bijective function (one-one correspondence)

inverse function.

Let f be a one to one correspondence from the set A to the set B. The inverse function of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a)=b$. The inverse function of f is denoted by f^{-1}, hence $f^{-1}(b) = a$ when $f(a) = b$.

composition of functions

Let $f: X \to Y$ and $g: Y \to Z$. Then the composition of f and g, denoted by gof is a function from X in to Z, ie, $gof: X \to Z$ defined by,

$$(gof)(x) = g[f(x)], \ \forall x \in X$$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "gof"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

Eg. consider the function $f: R \to R : f(x) = 2x+3$,

and $g: R \to R : g(x) = x^2$

Then $gof: R \to R$. and

$$gof(x) = g(f(x))$$
$$= g(2x+3)$$
$$= (2x+3)^2$$

2 Let f and g be the functions defined by $f: R \to R^+ \cup \{0\}$ with $f(x) = x^2$ and $g: R^+ \cup \{0\} \to R$ with $g(x) = \sqrt{x}$. what is the function $(fog)(x)$?

A). $fog: R^+ \cup \{0\} \to R^+ \cup \{0\}$ and

$$fog(x) = f(g(x))$$
$$= f(\sqrt{x})$$
$$= x$$

Note:

gof is defined only when $Im(f) \subseteq Dom(g)$. when gof is defined it is not necessary that fog is also defined. in general it is not necessary that $gof = fog$.